AN EMPIRICAL STUDY OF THE STABILITIES OF ESTIMATORS AND VARIANCE ESTIMATORS IN P.P.S. SAMPLING, II

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1. Introduction

In a previous paper (1967) we empirically investigated the stabilities of variance estimators jointly with the efficiencies of estimators of the population total for certain P.P.S. (Probability Proportional to Size) sampling methods for samples of n = 2 using actual finite populations and under the assumption of a super population model. In this paper after deriving the variance of the variance estimator in general and under the assumption of a super population model, we perform similar empirical studies as that mentioned above for samples of size n = 3 and 4. We also provide a computer computational scheme to calculate certain conditional probabilities for Murthy's (1957) method.

We have chosen only those methods (excepting one) which satisfy the following requirements:

a. a nonnegative, unbiased variance estimator should be available,

b. computations are feasible (timewise) on a high speed computer.

Based on these conditions we have selected the following methods for the present study:

1. The methods of Fellegi (1963), Carroll and Hartley (1964) and Sampford (1967), all using the Horvitz-Thompson (1952) estimator and satisfying $\Pi_j = np_j$ $(j = 1, \ldots, N)$, Π_j being the probability of including the j-th population unit in the sample.

2. The methods of Des Raj (1956) and Murthy (1957).

3. The method of Rao, Hartley, and Cochran (R.H.C.) (1962).

4. Lahiri's (1951) method.

The requirement a. is not satisfied by Lahiri's estimator. Nevertheless, we have included it in view of the recent work by Godambe (1966) based on concepts other than efficiency. Also, we exclude Fellegi's method for n = 4 because the computational cost becomes quite expensive due to calculation of the joint inclusion probabilities of any pair of units in the sample. Further the routine to calculate the working probabilities used to select the unit at the r-th draw, r = 1, ..., n often required several iterations for the populations we considered. Lahiri's method is excluded in our empirical study under the assumption of a super population model.

2. Formulae

In giving the formulae for the methods in this section we give the equations for any n and based on a super population model (Cochran(1946)) in which the finite population is regarded as being drawn from an infinite super population. The results obtained apply to the average of all finite populations that can be drawn from the super population. We assume the following, often used, super population model for the comparison of estimators:

$$y_i = \beta x_i + e_i \ , \ i = 1, \ldots, N$$
$$\epsilon(e_i|x_i) = 0 \ , \ \epsilon(e_i^2|x_i) = ax_i^g$$
$$\epsilon(e_ie_j|x_i,x_j) = 0 \ , \ a > 0 \ , \ g \geq 0$$

where ϵ denotes the average over all the finite populations that can be drawn from the super population. For the comparison of variance estimators we further assume that e_i's are normally distributed so that $\epsilon(e_i^4) = 3a^2x_i^{2g}$. In most practical situations, g is expected to lie between 1 and 2. Some theoretical results are available on the relative efficiencies of the estimators (Hanurav (1965), Rao (1966), and Vijayan (1967)) but no guidelines are available with regard to the relative magnitudes. Nothing is known on the stabilities of the variance estimators under the super population model.

Of course, the formulae we need for our empirical studies, and the computer programs, are those for n = 3 and 4.

The new formulae of this section are the Ev^2's while the other formulae were previously given in the references cited above. To check the formulae we considered the case when all the x-values are equal to one which is equivalent to simple random sampling. Under this condition all the formulae are identical except for Des Raj's method and were checked numerically. We also used a complete combinatorial evaluation to check the formulae for the R.H.C. method that is described in Appendix A.

2.1 Some IPPS (Inclusion Probabilities Proportional to Size) Methods Using the Horvitz-Thompson Estimator

The Horvitz-Thompson estimator of the population total, Y , for any n is

$$\hat{Y}_1 = \sum_{i=1}^{n} y_i/\Pi_i$$

where 1, 2, ..., n denote the units in the sample. For the methods of group (1) we have, since

$$\Pi_j = np_i = nx_i/\Sigma x_i \ ,$$

the Horvitz-Thompson estimators

$$\hat{Y}_1 = \sum_{i=1}^{n} y_i/np_i$$

with variance

$$V_1 = \sum\sum_{i<i'}^{N} (n^2p_ip_{i'}-\Pi_{ii'})(y_i/np_i-y_{i'}/np_{i'})^2$$

and variance estimator (due to Yates and Grundy (1953))

$$v_1 = \sum\sum_{i<i'}^{n} \left((n^2p_ip_{i'}-\Pi_{i'i'})/\Pi_{ii'}\right)(y_i/np_i-y_{i'}/np_{i'})^2$$

where $\Pi_{ii'}$ is probability of inclusion of units i and i' in the sample. Since Ev_1^2 is needed for

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the variance of the variance estimator, we write

$$Ev_1^2 = \sum_s \Pi(s)v_1^2(s)$$

$$= \sum_s \Pi(s)\left[\sum\sum_{\substack{i<i' \\ s\supset i,i'}}^{n} \left((n^2p_ip_{i'}-\Pi_{ii'})/\Pi_{ii'}\right)(y_i/np_i-y_{i'}/np_i)^2\right]^2$$

where $\sum_s$ denotes summation over all $\binom{N}{n}$ possible samples, s , $v_1(s)$ is v_1 for the sample s , $\Pi(s)$ the probability of obtaining the sample s . The formulae for $\Pi_{ii'}$ and $\Pi(s)$ for the various methods can be obtained from Fellegi (1963), Carroll and Hartley (1964), and Sampford (1967) or Bayless (1968).

Substituting the super population model into V_1 above and taking expectations we have

$$\epsilon V_1 = \epsilon\left[\sum_{i<i'}^{N}\sum (n^2p_ip_{i'}-\Pi_{ii'})(e_i/np_i-e_{i'}/np_{i'})^2\right]$$

$$= \epsilon\left[\sum_i^N (e_i^2/n^2p^2p_i^2)\left(np_i(1-np_i)\right) + \sum\sum_{i\neq i'}^{N} (\Pi_{ii'}-\Pi_i\Pi_{i'})(e_ie_{i'}/n^2p_ip_{i'})\right]$$

$$= aX^g/n \sum_{i=1}^{N} (1-np_i)p_i^{g-1}$$

which is independent of $\Pi_{ii'}$. Thus, all methods that use the Horvitz-Thompson estimator with $\Pi_i = np_i$ have the save average variance.

The evaluation of ϵEv_1^2 , v_1 being the Yates-Grundy variance estimator, is obtained as follows.

Since $Ev_1^2 = \sum_s \Pi(s)\ v_1^2(s)$, where $\Pi(s)$ and $v_1(s)$ are defined above, we have

$$\epsilon Ev_1^2 = \sum_s \Pi(s)\ \epsilon v_1{}'^2(s)$$

where $v_1{}'(s)$ is $v_1(s)$ with the super population model substituted into it.

Thus, it remains to evaluate $\epsilon v_1{}'^2(s)$, we

$$\epsilon v_1{}'^2(s) = \epsilon\left[\sum_{\substack{i<i' \\ s\supset ii'}}^{n}\sum H_{ii'}(e_i/np_i-e_{i'}/np_{i'})^2\right]^2$$

where

$$H_{ii'} = (n^2p_ip_{i'}-\Pi_{ii'})/\Pi_{ii'}$$

After taking expectations, and considerable manipulation, we have

$$\epsilon v_1{}'^2(s) = \left[\sum_{\substack{i<i' \\ s\supset ii'}}\sum_{} H_{ii'}K(g)_{ii'}\right]^2 + \sum_{\substack{i\neq i' \\ s\supset ii'}}^{n}\sum H_{ii'}^2K(g)_{ii'}^2 + \sum_{\substack{i\neq i'\neq i'' \\ s\supset ii'i''}}\sum\sum H_{ii'}H_{ii''}X^{2g}/n^4p_i^4 + \sum_{\substack{i\neq i'\neq i'' \\ s\supset ii'i''}}\sum\sum H_{ii'}H_{i'i''}X_{i'}^{2g}/n^4p_{i'}^4$$

where

$$K(g)_{ii'} = X_i^g/n^2p_i^2 + X_{i'}^g/n^2p_{i'}^2 \ .$$

2.2 The Des Raj Method

Des Raj proposed the uncorrelated unbiased estimators

$$t_1 = y_1'/p_1'$$

$$\cdots$$

$$t_r = \sum_{t-1}^{r-1} y_t' + y_r'(1-\sum_{t=1}^{r-1} p_t')/p_r'$$

$$\cdots$$

$$t_n = \sum_{t=1}^{n-1} y_t' + y_n'(1-\sum_{t=1}^{n-1} p_t')/p_n'$$

where (y_r', p_r') denote the y-value and the p-value of the unit selected at the r-th draw. As an unbiased estimator of Y we have

$$\hat{Y}_2 = \bar{t} = (1/n)\sum_{i=1}^{n} t_i$$

with

$$V_2 = V(\hat{Y}_2) = \frac{1}{2n^2}\sum_{i<i'}^{N}\sum p_ip_{i'}\left[1+\sum_{r=2}^{n} Q_{ii'}(r-1)\right](y_i/p_i - y_{i'}/p_{i'})^2$$

where $Q_{ii'}(r-1)$ denotes the probability of non-inclusion of unit i and i' in the first r-1 sample units and $Q_{ii}(0) = 1$ and variance estimator

$$v_2 = \Sigma(t_i-\bar{t})^2/n(n-1) \ .$$

Since the Q_{ii}'s of V_2 are very cumbersome to calculate, the formulae we used to calculate V_2 and Ev_2^2 using the above notation and letting $v_2(s')$ be v_2 for one of the n! possible orderings, s' , of the sample s , are

$$V_2 = \sum_{s}\sum_{s'} p_2(s')v_2(s')$$

and

$$Ev_2^2 = \sum_{s}\sum_{s'} p_2(s')v_2^2(s') .$$

where $\sum_s$ and $\sum_s$ denote the summation over all possible $\binom{N}{n}$ samples of size n and all possible n! orderings of a given sample, s , of size n respectively and

$$p_2(s') = p_{\ell_1}\left[p_{\ell_2}/(1-p_{\ell_1})\dots p_{\ell_n}/(1-p_{\ell_1}-\dots p_{\ell_{n-1}})\right]$$

$$(s' \supset \ell_t \quad t = 1, \dots, n) .$$

Substituting the super population into $v_2(s)$ to obtain $v_2'(s)$, we have $\epsilon V_2'$ and $\epsilon Ev_2'^2$ as defined as

$$\epsilon V_2' = \sum_{s}\sum_{s'} p_2(s') \ \epsilon v_2'(s')$$

and

$$\epsilon Ev_{22}'^2 = \sum_{s}\sum_{s'} p_2(s') \ \epsilon v_2'^2(s')$$

In appendix E we derive $\epsilon v_2'(s')$ and $\epsilon v_2'^2(s')$.

2.3 The Murthy Method

Murthy's estimator of Y for any n is

$$\hat{Y}_3 = \sum_{i=1}^{n} p(s|i)y_i/p(s)$$

where p(s) denotes the probability of getting the sample of n units; p(s|i) denotes the conditional probability of getting s given that unit "i" was drawn first (i = 1, 2, ..., n) .

Hence, for Murthy's selection procedure we have

$$p(s) = p_3(s) = \sum_{s'} p_3(s')$$

where $p_3(s')$ is the same as $p_2(s')$ of section 2.2 and $\sum_{s'}$ is defined in section 2.2. A scheme to calculate the p(s|i)'s is given in Appendix D.

The variance of Murthy's estimator is

$$V_3 = V(\hat{Y}_3) = (1/2) \sum_{i}\sum_{i'}^{N} \left\{(y_i/p_i-y_{i'}/p_{i'})^2 \right.$$
$$\left. p_ip_{i'}\left[1- \sum^*_{s\supset ii'} p(s|i)p(s|i')/p(s)\right]\right\}$$

where $\sum^*_{s\supset ii'}$ denotes summation over all sample s that contain units i and i' , with variance estimator

$$v_3 = v(\hat{Y}_3) = 1/2 \sum_{i\neq i}\sum^{n} \left\{p_ip_{i'}[p(s)p(s|ii') - p(s|i)\right.$$
$$\left. p(s|i')]/p^2(s)(y_i/p_i-y_{i'}/p_{i'})^2\right\}$$

where p(s|ii') denotes the conditional probability of s given that i and i' have been selected in the first 2 draws. In Appendix D we give a computing scheme to calculate the p(s|ii')'s .

Since V_3 involves the cumbersome sum $\sum^*_{s\supset ii'}$ we use a different formula to calculate V_3 which is easier to compute on a computer. It is

$$V_3 = \sum_s p_3(s)v_3(s)$$

where $v_3(s)$ is v_3 for the sample s . Also,

$$Ev_3^2 = \sum_s p_3(s)v_3^2(s) .$$

Substituting $y_i = \beta x_i + e_i$ into $v_3(s)$ to obtain $v_3'(s)$ we have ϵV_3 and ϵEv_3^2

$$\epsilon V_3 = \sum_s p_3(s) \ \epsilon v_3'(s)$$
$$\epsilon Ev_3^2 = \sum_s p_3(s) \ \epsilon v_3'^2(s) ,$$

where

$$\epsilon v_3(s) = \sum_{\substack{i < i' \\ s\supset ii'}}\sum^{n} M_{ii'} \ K(g)_{ii'}$$

with $K(g)_{ii'}$ defined in section 2.2 and

$$M_{ii'} = p_ip_{i'}[p(s)p(s|ii')-p(s|i)p(s|i')]/p^2(s)$$

and

$$\epsilon v_3'^2(s) = [\epsilon v_3'(s)]^2 + \sum_{\substack{i\neq i' \\ s\supset ii'}}\sum^{n} M_{ii'}^2K(g)_{ii'}^2$$

$$+ \sum_{\substack{i\neq i'\neq i'' \\ s\supset ii'i''}}\sum\sum M_{ii'}M_{ii''} \ x_i^{2g}/n^4p_i^4$$

$$+ \sum_{\substack{i\neq i'\neq i'' \\ s\supset ii'i''}}\sum\sum M_{ii'}M_{i'i''} \ x_{i'}^{2g}/n^4p_{i'}^4$$

2.4 The R.H.C. Method

The R.H.C. estimator of Y for any n is

$$\hat{Y}_4 = \sum_{i=1}^{n} y_iG_i/p_i$$

where $G_i = \sum_{Gr.i} p_t$ (i = 1, ..., n) and $\sum_{Gr.i}$ denotes summation over the p-values in random group i. The variance and variance estimator of $\hat{Y}_4$ are given by

$$V_4 = K\left(\sum_{t=1}^{N} y_t^2/np_t-Y^2/n\right)$$

where

$$K = n(\sum N_i^2-N)/N(N-1)$$

and

$$V_4 = W\left[\Sigma G_i(y_i/p_i-\hat{Y}_4)^2\right]$$

where

$$W = (\Sigma N_i^2-N)/(N^2-\Sigma N_i^2) \ .$$

In Appendix A a derivation of Ev_4^2 is given for any n. By substituting $y_i = \beta x_i + e_i$ into V_4 and taking the expectation, we have

$$\varepsilon V_4 = K\left[\Sigma x_t^g/p_t-\Sigma x_t^g\right]$$

where

$$K = (\Sigma N_i^2-N)/N(N-1)a \ .$$

A derivation of εEv_4^2 is given in Appendix C.

2.5 The Lahiri Method

Lahiri's estimator of Y for any n is

$$\hat{Y}_5 = \sum_{i=1}^{n} y_i / \sum_{i=1}^{n} p_i \ .$$

Using Lahiri's selection procedure it is easy to show that

$$p_5(s) = \sum_{s\supset i} x_i / \binom{N-1}{n-1} X$$

is the probability of obtaining the sample s. Hence, the variance of $\hat{Y}_5$ is

$$V_5 = \sum_s p_5(s)\ \hat{Y}_5^2(s) - Y^2$$

where $\hat{Y}_5(s)$ is $\hat{Y}_5$ for the sample s with variance estimator

$$v_5 = v_5(s) = \hat{Y}_5^2(s) - \left[\left(\sum_{s\supset i}^{n} y_i^2\right)/\binom{N-1}{n-1}p_5(s) + \left(\sum_{\substack{s\supset ii' \\ i\neq i'}}^{n}\sum y_i y_{i'}\right)/\binom{N-2}{n-2}p_5(s)\right]$$

where we note the expression in brackets is an unbiased estimate of Y^2 . Also

$$Ev_5^2 = \sum_s p_5(s)\ v_5^2(s) \ .$$

By the modified Lahiri variance estimator we mean

$$v_5' = \begin{cases} 0 & \text{when } v_5 \leq 0 \\ v_5 & \text{when } v_5 > 0 \end{cases} .$$

Thus, we have the mean square error as

$$\text{M.S.E.}(v'_5) = E(v_5'-V_5)^2 = Ev_5'^2 - 2V_5Ev_5' + V_5^2$$

where $Ev_5'^2$ and Ev_5' are easily obtained using $p_5(s)$ and the definition of v_5' .

2.6 The With Replacement Method

The customary estimator in unequal probability sampling and with replacement for any n is

$$\hat{Y}_6 = \frac{1}{n}\Sigma\, y_i/p_i$$

$$V_6 = \sum_{t=1}^{N} y_t^2/np_t - Y^2/n$$

and variance estimator

$$v_6 = \sum_{t=1}^{n} (y_t/p_t-\hat{Y}_6)^2/n(n-1) \ .$$

A derivation of Ev_6^2 for any n is given in Appendix B.

3. Empirical Results for n = 3

3.1 The Populations

We have chosen 14 natural populations for the empirical study described in Table 3.1. The first 12 populations were in the study for n = 2 (Rao and Bayless (1967)) with some of the populations sizes, N , reduced so as to reduce the amount of calculation. For example, we have included only one of the four Hanurav (1967) populations we considered in the n = 2 empirical study because they gave practically the same efficiencies for all methods. Similar reasoning was used for the omission of the other populations. We have added two natural populations from Yates' (1960) textbook, natural populations 13 and 14.

In observing Table 3.1 we see that the population sizes, N , range from 10 to 20, coefficient of variation of x, C.V.(x), range from .14 and 1.06, and correlations, ρ , from .50 to .99. Natural population 9 is different from the others in that it contains one large y/x ratio.

3.2 Stabilities of the Estimators

Table 3.2 gives the percent gains in efficiency of the estimators over Sampford's estimator for n = 3 (i.e., (V(Sampford's est.)/V(est.)-1) x 100) for the populations of Table 3.1. The following conclusions may be drawn:

1. The efficiencies of the Carroll-Hartley and Sampford estimators are about the same while Fellegi's estimator is consistently less efficient than either of them.
2. Des Raj's estimator is more efficient than the R.H.C. estimator, except for the small losses by populations 4, 5, 7, 9, and 12.
3. The loss in efficiency of Des Raj's estimator over Murthy's estimator is consistently small, with no differences in efficiencies for any given population being greater than five percentage points.
4. As with samples of size two, Lahiri's estimator is considerably more efficient than the other estimators when one or two units in the population have large sizes relative to the sizes of the remaining units, and samples containing these units have y-values that give good estimators of the population total Y(viz. populations 8 and 9). Otherwise, Lahiri's estimator has very poor efficiency compared with the other estimators. It looses to the customary estimator in p.p.s. sampling and with replacement in 5 of the 14 populations.
5. Murthy's estimator is consistently more efficient than the R.H.C. estimator except for small loss of one percentage point for population 9.
6. Murthy's estimator is slightly better than those of Sampford, Carroll-Hartley, and

Fellegi, the only loss of any magnitude being for population one.

3.3 Stabilities of the Variance Estimators

The measure of stability used to compare the variance estimators under study is $(C.V.^2(\text{Sampford's Var. Est.})/C.V.^2(\text{Var. Est.})-1) \times 100$. Table 3.3 gives the percent gains in efficiency of the variance estimators over Sampford's variance estimator for the populations listed in Table 3.1. The conclusions we draw are as follows:

1. Lahiri's modified variance estimator is consistently, and considerably, less efficient than the other variance estimators in the study except for population 9 with the 'wild' y/x ratio where it has a striking 71 percent gain over Sampford's estimator.
2. Stabilities of the Sampford, Carroll-Hartley, and Fellegi variance estimators are essentially the same for all populations.
3. The variance estimators of Murthy, Des Raj, and R.H.C. are consistently better than the "with replacement" estimator.
4. The R.H.C. variance estimator is more efficient than the Sampford, Carroll-Hartley, Fellegi, and Des Raj variance estimators for all populations excepting populations 5 and 6.
5. Murthy's variance estimator is consistently more efficient than Des Raj's variance estimator; however, the gains are small. Murthy's and Des Raj's variance estimators are more efficient than those of Sampford, Carroll-Hartley, and Fellegi excepting for population 1.
6. The R.H.C. variance estimator is often more efficient than Murthy's variance estimator.

3.4 Stabilities of the Estimators under the Assumption of the Super Population Model

Table 3.4 gives the percent gains in average efficiency of the estimators over Sampford's estimator (i.e., $(\epsilon V(\text{Sampford's est.})/\epsilon V(\text{est.})-1) \times 100$) for the populations of Table 3.1 for g = 1.50, 1.75, and 2.00. Since g is usually expected to be ≥ 1.5, we have not included values of $g < 1.5$ to save computer time. The conclusions we draw are as follows:

1. As in the case n = 2, Murthy's estimator is always more efficient than the Horvitz-Thompson estimator for $g \leq 1.75$ over all populations. For g = 2 Murthy's estimator looses by no more than two percentage points to the Horvitz-Thompson estimator.
2. For all values of g considered, Murthy's estimator is consistently more efficient than the R.H.C. estimator. For some populations the gains are considerable.
3. Des Raj's estimator is less efficient than the Sampford estimator for $g \geq 1.5$.
4. Des Raj's estimator is more efficient than the R.H.C. estimator for all populations and all values of g except for populations 5 and 7 where a loss of less than one percentage point results.

3.5 Stabilities of the Variance Estimators under Assumption of the Super Population Model

The most appropriate measure of the stability of a variance estimator v under the assumption of the super population model appears to be $\epsilon[C.V.^2(v)]$, i.e., average $(C.V.)^2$ of the variance estimator. However, since $\epsilon[C.V.^2(v)]$ is the expectation of the ratio of two random variables, the evaluation is difficult. We have, therefore, used the alternative measures

$$\frac{\epsilon E[v-\epsilon V]^2}{(\epsilon V^2)} = \frac{\epsilon[Ev^2] - (\epsilon V)^2}{(\epsilon V)^2}$$

which is readily evaluated. Notice that this measure actually measures the variability of v around the average variance ϵV . We could also have considered the measure $\epsilon V(v)/(\epsilon V)^2$. We, however, expect that the measures $\epsilon V(v)/(\epsilon V)^2$ and $\epsilon[C.V.^2(v_i)]$ would lead to similar conclusions. It will be seen that the above measures are independent of β for all the methods considered here.

Using the stability measure above, we present in Table 3.5 the percent gains in average efficiency of the variance estimators over Sampford's estimator for g-values of 1.50, 1.75, and 2.00 for the populations of Table 3.1. The conclusions we draw are as follows:

1. The R.H.C. variance estimator is consistently more efficient than Murthy's variance estimator for $g \leq 1.75$, except for g = 1.50 and population 7. However, for g = 2, Murthy's variance estimator is consistently more efficient than the R.H.C. variance estimator.
2. The absolute percent differences between Murthy's variance estimator stability and Des Raj's variance estimator stability is less than or equal to 3 percent for all populations except population 4 for $g \leq 1.75$, and populations 4 and 8 for $g = 2.00$. Thus, these variance estimators are of about the same stability.
3. The R.H.C. and Murthy's variance estimators are consistently more efficient than Sampford's variance estimator as well as Carroll-Hartley, and Fellegi's variance estimator for all g. The Des Raj variance estimator is also consistently more efficient except for a few very small losses. The gains are appreciable for several of the populations.
4. The stabilities of Carroll-Hartley, and Sampford variance estimators are practically identical for all values of g.
5. Fellegi's variance estimator is consistently less efficient compared to the Sampford variance estimator for all values of g. For some of the populations, the losses are large.

4. Empirical Results for n = 4

4.1 The Populations

For this empirical study, we have selected 10 populations out of the 14 populations used for n = 3, with some of the populations sizes, N , decreased for computational reasons with the units making up the populations selected at random.

Table 4.1 describes the 10 populations where we see that N ranges from 10 to 16, C.V.(X) from .14 to 1.06, and ρ from .65 to .99. Population 7 corresponds to population 9 of Table 3.1 for n = 3 in that it contains the one large y/x ratio.

4.2 Stabilities of the Estimators

The percent gains in efficiency of the estimators over Sampford's estimator for n = 4, for the populations listed in Table 4.1, are given in Table 4.2. The conclusions that we draw are as follows:

1. The efficiencies of the Carroll-Hartley and the Sampford estimators are practically the same.
2. Murthy's estimator is consistently more efficient than the R.H.C. estimator.
3. The losses of the Des Raj estimator over Murthy's estimator are large for certain populations; in particular, a difference of at least 7 percentage points exists for populations 3, 4, 6, 7, and 9. A reason for this is that the n/N is large, at least 25%, and/or C.V.(X) is large. Although Pathak (1967) proved that if N is large compared to n , the variance of the Des Raj and Murthy estimators are identical to O(N'). It appears N is not large enough for his theory to apply in this case.
4. The Des Raj estimator appears slightly more efficient than the R.H.C. estimator.
5. In 4 out of the 10 populations, Lahiri's estimator has better efficiency than the other estimators, viz. populations 2, 4, 6, 7. The apparent reason for this is due to the small C.V.(X)-values for populations 2, 4, and 6 and the large variations in the y/x values for population 7. Since we know that, as C.V.(X)$\to$0 , all these methods tend to equal probability sampling without replacement; we would expect Lahiri's estimator to have good efficiency for small C.V.(X)-values. For most of the other populations, Lahiri's estimator has very poor efficiency compared to the other estimators. The customary estimator in p.p.s. sampling and with replacement has better efficiency than Lahiri's estimator for populations 3 and 5.

As with n = 2 and 3, it still appears that Murthy's estimator compares favorably with the Carroll-Hartley and Sampford estimators.

4.3 Stabilities of the Variance Estimators

The measure of stability used to compare the variance estimators is the same as that used for n = 3 in section 3.3. Table 4.3 gives the percent gains in efficiency of the variance estimators over Sampford's variance estimator for the populations of Table 4.1. The conclusions that we draw are as follows:

1. The Carroll-Hartley variance estimator and Sampford's variance estimator are essentially the same with regard to stability.
2. The variance estimators of Murthy, Des Raj, and R.H.C. are consistently more efficient than Sampford's variance estimator or the customary variance estimator in sampling with replacement.
3. Murthy's variance estimator is more efficient than the Des Raj variance estimator excepting that for populations 8 and 9 the latter is slightly better.
4. The R.H.C. variance estimator is more efficient than Murthy's variance estimator except for the small losses for populations 1 and 5.
5. The Modified Lahiri variance estimator has essentially the same efficiency as the unbiased Lahiri variance estimator except for population 7 which has the one large y/x ratio. These variance estimators are considerably less efficient than are the other variance estimators.

4.4 The Populations under the Assumption of the Super Population Model

Table 4.4 gives the populations for the empirical study under consideration. For all populations we have chosen the population sizes, N , to be 10 with C.V.(x) values ranging from .14 to .82 and ρ from .34 to .99. The populations were selected at random from the original populations subject to the restriction that the condition $\Pi_i \leq 4p_i$ is satisfied. Of course, the reason for choosing N = 10 for all the populations is due to computational costs. It should be pointed out that, due to the sampling fraction of 40%, comparisons of efficiency from n = 3 to n = 4 are not meaningful.

4.5 Stabilities of the Estimators under the Assumption of the Super Population Model

The Table 4.5 gives the percent gains in average efficiency of the estimators over Sampford's estimator for the populations of Table 4.4 for g = 1.5, 1.75, and 2.00. The following conclusions are drawn:

1. Except for the one percent loss in efficiency for populations 4 and 6 with small C.V.(y) and C.V.(x), Des Raj's estimator is slightly more efficient than the R.H.C. estimator for all g-values and populations.
2. As with n = 2 and 3, Murthy's estimator is more efficient than Sampford's estimator for all populations when g < 1.75. For g = 2, Murthy's estimator looses 8 and 9 percentage points in efficiency to Sampford's estimator for the populations 3 and 5 with highest C.V.(y) and C.V.(x).
3. The gains in efficiency of Murthy's estimator over Des Raj's estimator are considerable, expecially for the populations with large C.V.(x). A reason for this is that the sampling fraction is large for this study.

4.6 Stabilities of the Variance Estimators under the Assumption of the Super Population Model

Using the stability measures defined in section 3.5, Table 4.6 gives the percent gains in average efficiency for g = 1.50, 1.75, and 2.00 for the populations of Table 4.4. The following conclusions are drawn:

1. For all g-values and populations the stability of Carroll-Hartley variance estimator and the Sampford variance estimators are practically the same except for population 3 with a large C.V.(x) value.
2. For g = 1.5 Murthy's variance estimator is less efficient than the R.H.C. variance

estimator, except for the small gains for populations 4 and 6. For g = 1.75 it is not clear cut because Murthy's variance estimator is more efficient for 4 of the 10 populations with the losses much smaller than that in the case of n = 3. For g = 2.0, Murthy's variance estimator is consistently more efficient than the R.H.C. estimator for all populations. This conclusion agrees with our results for n = 3 but not with those for n = 2.

3. Murthy's variance estimator is consistently more efficient than Des Raj's variance estimator for all g- values and all populations.
4. The variance estimators of Murthy, Des Raj, and R.H.C. are consistently more efficient than those of Carroll-Hartley and Sampford for all populations and g-values except for the very small losses of the Des Raj's variance estimator for populations 6 (with the g = 2). The gains in efficiency are considerable for several of the populations, especially for those with moderate or large C.V.(x).

5. Overall Conclusions

The following overall (n = 3, and 4) conclusions may be drawn from our studies:

1. It appears the results under the super population model are in agreement with those from the empirical study using the actual y-data.
2. The I.P.P.S. sampling methods using the Horvitz-Thompson estimator are practically the same with respect to efficiencies of estimating the population total and the stabilities of variance estimation.
3. Murthy's method is preferable over the other methods when a stable estimator as well as a stable variance estimator are required.
4. The R.H.C. variance estimator is fairly stable, but the R.H.C. estimator might lead to significant losses in efficiency.

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APPENDIX A

A derivation of Ev_4^2 for the R.H.C. method for any n

The R.H.C. sampling scheme involves splitting the population at random into n groups of sizes $N_1, \ldots, N_n$ where $N_1 + \ldots + N_n = N$. Then draw a sample of size one with probabilities proportional to $p_t = x_t/\Sigma x_t$ from each of these n groups independently. Thus, if the t-th unit falls in group i, the probability that it will be selected is p_t/G_i where $G_i = \sum_{\text{Group } i} p_t$. With this 'set up'

$$Y_4 = \sum_{i=1}^{n} y_i \frac{G_i}{p_i}$$ is an unbiased estimate of Y with variance $V_4 = k_1 V_6$ where $k_1 = n(\Sigma N_i^2 - N)/N(N-1)$ and V_6 is the well-known formula for the variance of the customary estimator in sampling with unequal probabilities and with replacement. Also

$$v_4 = k_2 \Sigma G_i \left(\frac{y_i}{p_i} - \hat{Y}_4\right)^2$$, can be shown to be an unbiased estimate of V_4, where

$$k_2 = (\Sigma N_i^2 - N)/(N^2 - \Sigma N_i^2) .$$

Thus, in order to derive $V(v_4)$ we need to determine Ev_4^2 . This will be done by using the familiar conditional expectation argument. Let E_2 denote the expectation for a given split of the population and E_1 the expectation over all possible splits of the population into n groups of sizes $N_1, \ldots, N_n$. Therefore $Ev_4^2 = E_1 E_2 v_4^2$.

Now

$$v_4^2 = k_2^2 \Big[(\Sigma G_i y_i^2/p_i^2)^2 - 2(\Sigma G_i y_i^2/p_i^2)(\Sigma G_i y_i/p_i)^2 + (\Sigma y_i G_i/p_i)^4\Big]$$

and taking expectation with respect to E_2 term by term of v_4^2 and introducing the indicator random variable

$$t_{i\ell} = \begin{cases} 1 \text{ if the } \ell\text{-th unit falls in the i-th group} \\ 0 \text{ otherwise} \\ i = 1, \ldots, n \text{ and } \ell = 1, \ldots, N \end{cases}$$

where

$$\sum_{i=1}^{n} t_{i\ell} = 1 \qquad \sum_{\ell=1}^{N} t_{i\ell} = N_i$$

and letting

$$C_{inm} = \sum_{\ell=1}^{N} t_{i\ell} y_\ell^n / p_\ell^m$$

we have

$$D_1(\underline{Y},\underline{p},\underline{T}) = E_2(\Sigma G_i y_i^2/p_i^2)^2 = \sum_{i=1} C_{i,4,3}\, C_{i,0-1} + \sum\sum_{i\neq i'}^{n} C_{i,2,1}\, C_{i',21}$$

$$D_2(Y,p,T) = E_2(\Sigma G_i y_i^2/p_i^2)(\Sigma G_i y_i/p_i)^2 = \sum_{i=1}^{n} C_{i,4,3}\, C_{i,0,-1}^2 + \sum\sum_{i\neq i'}^{n} C_{i,2,1}\, C_{i',2,1}\, C_{i',0,-1} + 2\sum\sum_{i\neq i'}^{n} C_{i,0,-1} C_{i,3,2} C_{i',1,0} + \sum\sum\sum_{i\neq i'\neq i''}^{n} C_{i,2,1} C_{i',1,0} C_{i'',1,0}$$

$$D_3(\underline{Y},\underline{p},T) = E_2\Big(\sum_{i=1}^{n} y_i G_i/p_i\Big)^4 = \sum_{i=1}^{n} C_{i,0,-1}^3 C_{i,4,3} + 3\sum\sum_{i\neq i'}^{n} C_{i,0,-1} C_{i,2,1} C_{i',2,1} C_{i',0,-1} + 4\sum\sum_{i\neq i'}^{n} C_{i,0,-1}^2 C_{i,3,2} C_{i',1,0} + 6\sum\sum\sum_{i\neq i'\neq i''}^{n} C_{i,0,-1} C_{i,2,1} \cdot C_{i,1,0} C_{i'',1,0} + \sum\sum\sum\sum_{i\neq i'\neq i''\neq i'''}^{n} C_{i,1,0} C_{i',1,0} \cdot C'_{i'',1,0} C_{i''',1,0}$$

where $\underline{Y} = [y_1, \ldots, y_N]$, is N x 1 vector of y-values
$\underline{p} = [p_1, \ldots, p_N]$, is N x 1 vector of p-values
$T = \{t_{i\ell}\}$ is n x N matrix of the indicator random variables $t_{i\ell}$. Hence,

$$Ev_4^2 = E_1 D_1(\underline{Y},\underline{p},T) - 2E_1 D_2(\underline{Y},\underline{p},T) + E_1 D_3(\underline{Y},\underline{p},T).$$

Taking expectation using E_1 is quite involved and before doing so we might point that an easy combitorial solution to evaluating Ev_4^2 would be to enumerate all the possible

$$W = \frac{N!}{N_1! \ldots N_n!} = \binom{N}{N_1}\binom{N-N_1}{N_2} \cdots \binom{N-N_1 \ldots N_n}{N_n}$$

distinct T matrices, each one corresponding to a different 'split' of the population, and for each

T evaluate $D_1(\underline{Y},\underline{p},T)$, $D_2(\underline{Y},\underline{p},T)$, and $D_3(\underline{Y},\underline{p},T)$ and then take the simple average of all W evaluations. Of course, this solution is much simpler than taking E_1 but the computation becomes out of practical reach when either n or N is large. For small values of n and N this approach was used to check on the solution below.

To evaluate $E_1D_j(\underline{Y},\underline{p},T)$ j = 1, 2, 3 the following notation and conditional expectation or probabilities are needed.

Letting $A_{ij} = \sum_{\ell=1}^{N} y_\ell^i/p_\ell^j$ and $b_{ijk} = \frac{N_i-j}{N-k}$

and from equal probability sampling and without replacement we have

$$E_1(t_{i\ell}) = p_r(t_{i\ell}=1) = N_i/N = b_{i,0,0}$$

$$E_1(t_{i\ell}t_{i\ell'}) = b_{i,0,0}b_{i,1,1}$$

$$E_1(t_{i\ell}t_{i\ell'}t_{i\ell''}) = b_{i,0,0}b_{i,1,1}b_{i,2,2}$$

$$E_1(t_{i\ell}t_{i\ell'}t_{i\ell''}t_{i\ell'''}) = b_{i00}b_{i11}b_{i22}b_{i33}$$

$$E_1(t_{i\ell}t_{i'\ell'}) = b_{i00}b_{i'01}$$

$$E_1(t_{i\ell}t_{i\ell'}t_{i'\ell''}) = b_{i,0,0}b_{i,1,1}b_{i',0,2}$$

$$E_1(t_{i\ell}t_{i'\ell'}t_{i''\ell''}) = b_{i,0,0}b_{i',0,1}b_{i'',0,2}$$

$$E_1(t_{i\ell}t_{i'\ell'}t_{i'\ell''}) = b_{i00}b_{i'01}b_{i'12}$$

$$E_1(t_{i\ell}t_{i'\ell'}t_{i\ell''}) = b_{i00}b_{i'01}b_{i12}$$

$$E_1(t_{i\ell}t_{i\ell'}t_{i'\ell''}t_{i''\ell'''}) = b_{i00}b_{i11}b_{i'02}b_{i''03}$$

$$E_1(t_{i\ell}t_{i'\ell}t_{i\ell'}t_{i'\ell''}) = b_{i00}b_{i'01}b_{i12}b_{i'13}$$

$$E_1(t_{i\ell}t_{i\ell'}t_{i\ell''}t_{i'\ell'''}) = b_{i00}b_{i11}b_{i22}b_{i'03}$$

$$E_1(t_{i\ell}t_{i'\ell'}t_{i''\ell''}t_{i'''\ell'''}) = b_{i00}b_{i01}b_{i''02}b_{i'''03}$$

Defining,

$$F_{1,1}(\underline{A}) = A_{42}$$

$$F_{1,2}(\underline{A}) = A_{43}A_{0-1} - A_{42}$$

$$F_{2,5}(\underline{A}) = A_{21}^2 - A_{42}$$

$$F_{3,1}(\underline{A}) = -2A_{41}$$

$$F_{3,2}(\underline{A}) = -2A_{43}A_{0-2} - 4A_{42}A_{0-1} + 6A_{41}$$

$$F_{3,3}(\underline{A}) = -2A_{43}A_{0-1}^2 + 2A_{43}A_{0-2} + 4A_{42}A_{0-1} - 4A_{41}$$

$$F_{4,5}(\underline{A}) = -2A_{21}A_{20} + 2A_{41}$$

$$F_{4,6}(\underline{A}) = -2A_{21}^2A_{0-1} + 4A_{21}A_{20} + 2A_{42}A_{0-1} - 4A_{41}$$

$$F_{5,5}(\underline{A}) = -4A_{31}A_{10} + 4A_{41}$$

$$F_{5,7}(\underline{A}) = -4A_{32}A_{10}A_{0-1} + 4A_{42}A_{0-1} + 4A_{31}A_{10} + 4A_{32}A_{1-1} - 8A_{41}$$

$$F_{6,11}(\underline{A}) = -2A_{21}A_{10}^2 + 2A_{21}A_{20} + 4A_{31}A_{10} - 4A_{41}$$

$$F_{7,1}(\underline{A}) = A_{40}$$

$$F_{7,2}(\underline{A}) = 3A_{41}A_{0-1} + 3A_{42}A_{0-2} + A_{43}A_{0-3} - 7A_{40}$$

$$F_{7,3}(\underline{A}) = 3A_{42}A_{0-1}^2 - 6A_{42}A_{0-2} - 9A_{41}A_{0-1} + 3A_{43}A_{0-2}A_{0-1} - 3A_{43}A_{0-3} + 12A_{40}$$

$$F_{7,4}(\underline{A}) = A_{43}A_{0-1}^2 - 3A_{43}A_{0-2}A_{0-1} + 2A_{43}A_{0-3} - 3A_{42}A_{0-1}^2 + 3A_{42}A_{0-2} + 6A_{41}A_{0-1} - 6A_{40}$$

$$F_{8,5}(\underline{A}) = 3A_{20}^2 - 3A_{40}$$

$$F_{8,6}(\underline{A}) = 3A_{20}A_{21}A_{0-1} - 3A_{20}^2 - 3A_{2-1}A_{21} - 3A_{41}A_{0-1} + 6A_{40}$$

$$F_{8,8}(\underline{A}) = 3A_{0-1}^2A_{21}^2 - 3A_{0-1}^2A_{42} - 3A_{0-2}A_{21}^2 + 3A_{0-2}A_{42} + 6A_{20}^2 - 12A_{20}A_{21}A_{0-1} + 12A_{2-1}A_{21} + 12A_{41}A_{0-1} - 18A_{40}$$

$$F_{8,10}(\underline{A}) = F_{8,6}(\underline{A})$$

$$F_{9,5}(\underline{A}) = 4A_{30}A_{10} - 4A_{40}$$

$$F_{9,7}(\underline{A}) = 8A_{31}A_{0-1}A_{10} - 8A_{31}A_{1-1} - 8A_{0-1}A_{41}$$

$$+ 4A_{0-2}A_{32}A_{10} - 4A_{0-2}A_{42} - 12A_{30}A_{10}$$

$$- 4A_{32}A_{1-2} + 24A_{40}$$

$$F_{9,9}(\underline{A}) = 4A_{32}A_{10}A_{0-1}^2 - 4A_{42}A_{0-1}^2 - 4A_{32}A_{10}A_{0-2}$$

$$+ 4A_{42}A_{0-2} - 8A_{31}A_{10}A_{0-1} + 8A_{31}A_{1-1}$$

$$+ 8A_{30}A_{10} + 16A_{41}A_{0-1} - 24A_{40}$$

$$- 8A_{32}A_{1-1}A_{0-1} + 8A_{1-2}A_{32}$$

$$F_{10,11}(\underline{A}) = 6A_{20}A_{10}^2 - 6A_{20}^2 - 12A_{30}A_{10} + 12A_{40}$$

$$F_{10,12}(\underline{A}) = 6A_{21}A_{10}^2A_{0-1} - 6A_{21}A_{20}A_{0-1} - 6A_{20}A_{10}^2$$

$$+ 6A_{20}^2 + 12A_{1-1}A_{31} - 12A_{21}A_{1-1}A_{10}$$

$$+ 24A_{30}A_{10} - 12A_{31}A_{10}A_{0-1}$$

$$+ 12A_{21}A_{2-1} + 12A_{41}A_{0-1} - 36A_{40}$$

$$F_{11,13}(\underline{A}) = A_{10}^4 - 6A_{20}A_{10}^2 + 3A_{20}^2 + 8A_{30}A_{10} - 6A_{40}$$

$$F_{12,1}(\underline{A}) = A_{30}$$

$$F_{12,2}(\underline{A}) = A_{32}A_{0-2} + 2A_{31}A_{0-1} - 3A_{30}$$

$$F_{12,3}(\underline{A}) = A_{32}A_{0-1}^2 - A_{32}A_{0-2} - 2A_{31}A_{0-1} + 2A_{30}$$

$$F_{12,5}(\underline{A}) = 3A_{20}A_{10} - 3A_{30}$$

$$F_{12,6}(\underline{A}) = 3A_{21}A_{10}A_{0-1} - 3A_{20}A_{10} - 3A_{31}A_{0-1}$$

$$- 3A_{21}A_{1-1} + 6A_{30}$$

$$F_{12,11}(\underline{A}) = A_{10}^3 - 3A_{20}A_{10} + 2A_{30}$$

If $F_{k,j}(\underline{A})$ (k=1,...12 and j=1,...,13) is not given it is assumed to be zero, and

$$d_1 = \sum_{i=1}^{n} b_{i00}$$

$$d_2 = \sum_{i=1}^{n} b_{i00}b_{i11}$$

$$d_3 = \sum_{i=1}^{n} b_{i00}b_{i11}b_{i22}$$

$$d_4 = \sum_{i=1}^{n} b_{i00}b_{i11}b_{i22}b_{i33}$$

$$d_5 = \sum\sum_{i\neq i'}^{n} b_{i00}b_{i'01}$$

$$d_6 = \sum\sum_{i\neq i'}^{n} b_{i00}b_{i'01}b_{i'12}$$

$$d_7 = \sum\sum_{i\neq i'}^{n} b_{i00}b_{i11}b_{i'02}$$

$$d_8 = \sum\sum_{i\neq i'}^{n} b_{i00}b_{i12}b_{i'01}b_{i'03}$$

$$d_9 = \sum\sum_{i\neq i'}^{n} b_{i00}b_{i11}b_{i22}b_{i'03}$$

$$d_{10} = \sum\sum_{i\neq i'}^{n} b_{i00}b_{i12}b_{i'01}$$

$$d_{11} = \sum\sum\sum_{i\neq i'\neq i''}^{n} b_{i00}b_{i'01}b_{i''02}$$

$$d_{12} = \sum\sum\sum_{i\neq i'\neq i''}^{n} b_{i00}b_{i11}b_{i'02}b_{i''03}$$

$$d_{13} = \sum\sum\sum\sum_{i\neq i'\neq i''\neq i'''}^{n} b_{i00}b_{i'01}b_{i''02}b_{i'''03}$$

and

$$S_k = \sum_{j=1}^{13} F_{k,j}(A)d_j \quad k = 1,\ldots,12$$

we have

$$E_1D_1(\underline{Y},\underline{P},T) = \sum_{k=1}^{2} S_k$$

$$-2E_1D_2(\underline{Y},\underline{P},T) = \sum_{k=3}^{6} S_k$$

$$E_1D_3(\underline{Y},\underline{P},T) = \sum_{k=7}^{11} S_k$$

and

$$E_1v_4^2 = \sum_{k=1}^{11} S_k \ .$$

APPENDIX B

A derivation of Ev_6^2 for unequal probabilities and with replacement method for any n

From the characteristic function of the multinomial distribution we obtain the following needed moments:

$$Et_i = np_i$$

$$Et_i^2 = n(n-1)p_i^2 + np_i$$

$$Et_i t_{i'} = n(n-1)p_i p_{i'}$$

$$Et_i t_{i'} t_{i''} = np_i(n-1)p_{i'}(n-2)p_{i''}$$

$$Et_i^2 t_{i'} = n(n-1)(n-2)p_i^2 p_{i'} + n(n-1)p_i p_{i'}$$

$$Et_i^4 = n(n-1)(n-2)(n-3)p_i^4 + 6n(n-1)(n-2)p_i^3 + 7n(n-1)p_i^2 + np_i$$

$$Et_i^2 t_{i'}^2 = n(n-1)(n-2)(n-3)p_i^2 p_{i'}^2 + n(n-1)(n-2)p_i^2 p_{i'} + n(n-1)(n-2)p_i p_{i'}^2 + n(n-1)p_i p_{i'}$$

$$Et_i^3 t_{i'} = n(n-1)(n-2)(n-3)p_i^3 p_{i'} + 3n(n-1)(n-2)p_i^2 p_{i'} + n(n-1)p_i p_{i'}$$

$$Et_i^2 t_{i'} t_{i''} = n(n-1)(n-2)(n-3)p_i^2 p_{i'} p_{i''} + n(n-1)(n-2)p_i p_{i'} p_{i''}$$

$$Et_i t_{i'} t_{i''} t_{i'''} = n(n-1)(n-2)(n-3)p_i p_{i'} p_{i''} p_{i'''}$$

Writing v_6 as

$$v_6 = 1/n(n-1)\left[\Sigma t_i y_i^2/p_i^2 - n\,\hat{Y}_6^2\right]$$

where

$$\hat{Y}_6 = \Sigma t_i y_i / np_i$$

$$t_i = \begin{cases} 1 & \text{if i-th unit is in the sample} \\ 0 & \text{otherwise} \end{cases}$$

using the above moments of the multinomial distribution we have,

$$Ev_6^2 = \left[1/n(n-1)\right]^2 E\left[\left(\sum_{i=1}^{N} t_i y_i^2/p_i^2\right)^2 - 2n\hat{Y}_6^2 \sum_{i=1}^{N} t_i y_i^2/p_i^2 + n^2\hat{Y}_6^4\right]$$

where

$$E\left(\Sigma t_i y_i^2/p_i^2\right)^2 = nA_{43} + n(n-1)A_{21}^2$$

$$n^2E\left(\hat{Y}_6^2\,\Sigma t_i y_i^2/p_i^2\right) = nA_{43} + n(n-1)A_{21}^2 + 2\,n(n-1)A_{10}A_{32} + n(n-1)(n-2)A_{21}A_{10}^2$$

$$n^4E\hat{Y}_6^4 = n(n-1)(n-2)(n-3)A_{10}^4 + 6n(n-1)(n-2)A_{10}^2A_{21} + 4n(n-1)A_{32}A_{10} + 3n(n-1)A_{21}^2 + nA_{43}$$

$$A_{ij} = \sum_{t=1}^{N} y_t^i/p_t^j$$

APPENDIX C

A derivation of ϵEv_4^2 for the R.H.C. method for any n.

Substituting $y_i = \beta x_i + e_i$ into Ev_4^2, which involves replacing each y_i by e_i since it is since it is easily shown that Ev_4^2 is independent of the βx_i term, we have after taking expectation,

$$\epsilon Ev_4^2 = \sum_{k=1}^{8} R_k$$

where

$$R_k = \sum_{j=1}^{13} H_{k,j}(B)d_j \qquad k = 1, \ldots, 8 .$$

Where the d_j's are given in Appendix A and

$$H_{1,1}(\underline{B}) = B_{42}$$

$$H_{1,2}(\underline{B}) = B_{43}\, B_{0-1} - B_{42}$$

$$H_{2,5}(\underline{B}) = B_{21}^2 - (1/3)B_{42}$$

$$H_{3,1}(\underline{B}) = -2B_{41}$$

$$H_{3,2}(\underline{B}) = -2B_{43}\, B_{0-2} - 4B_{42}B_{0-1} + 6B_{41}$$

$$H_{3,3}(\underline{B}) = -2B_{43}\, B_{0-1}^2 + 2B_{43}\, B_{0-2} + 4B_{42}\, B_{0-1} - 4B_{41}$$

$$H_{4,5}(\underline{B}) = -2B_{21}\, B_{20} + (2/3)B_{41}$$

$$H_{4,6}(\underline{B}) = (2/3)B_{42}\, B_{0-1} - 2B_{0-1}\, B_{21}^2 - (4/3)B_{41} + 4B_{21}\, B_{20}$$

$$H_{7,1}(\underline{B}) = B_{40}$$

$$H_{7,2}(\underline{B}) = 3B_{41}\, B_{0-1} + 3B_{42}\, B_{0-2} + B_{43}\, B_{0-3} - 7B_{40}$$

$$H_{7,3}(\underline{B}) = 3B_{42}\, B_{0-1}^2 - 6B_{42}\, B_{0-2} - 9B_{41}\, B_{0-1} + 3B_{43}\, B_{0-2}\, B_{0-1} - 3B_{43}\, B_{0-3} + 12B_{40}$$

$$H_{7,4}(\underline{B}) = B_{43}\, B_{0-1}^2 - 3B_{43}\, B_{0-2}\, B_{0-1} + 2B_{43}\, B_{0-3} - 3B_{42}\, B_{0-1}^2 + 3B_{42}\, B_{0-2} + 6B_{41}\, B_{0-1} - 6B_{40}$$

$$H_{8,5}(\underline{B}) = 3B_{20}^2 - B_{40}$$

$$H_{8,6}(\underline{B}) = 3B_{0-1}\, B_{21}\, B_{20} + 2B_{40} - 3B_{20}^2 - 3B_{21}\, B_{2-1} - B_{41}\, B_{0-1}$$

$$H_{8,8}(\underline{B}) = 3B_{0-1}^2\, B_{21}^2 - B_{0-1}^2\, B_{42} - 3B_{0-2}\, B_{21}^2 + B_{0-2}\, B_{42}$$

$$+ 6B_{20}^2 - 12B_{0-1}\, B_{21}\, B_{20} + 12B_{21}\, B_{2-1} + 4B_{41}\, B_{0-1} - 6B_{40}$$

$$H_{8,10}(\underline{B}) = H_{8,6}(\underline{B})$$

Where if $H_{k,j}(\underline{B})$ ($k = 1, \ldots 8$ and $j = 1, \ldots 13$) is not given it is zero, and

$$B_{0j} = \sum_{t=1}^{N} 1/p_t^j$$

$$B_{2j} = \sum_{t=1}^{N} x_t^g/p_t^j$$

$$B_{4j} = \sum_{t=1}^{N} 3x_t^{2g}/p_t^j$$

APPENDIX D

A scheme to calculate the conditional probabilities $p(s|i)$ and $p(s|ii')$ for Murthy's method

Scheme:

1. Rearrange the integers of a given sample s, $\ell_1 \ldots \ell_n$, $1 \le \ell_i \le N$ such that $\ell_1 < \ell_2 < \ldots \ell_n$.
2. Form n groups of the n! permutations by placing ℓ_i as the first element of the (n-1)! permutations in group i.
3. Within group i, i = 1, ..., n, form n-1 subgroups by placing $\ell_{i'}$, $i' \neq i$, as the second element of the (n-2)! permutations in subgroup i' with the elements with subgroups arranged by the ascending order of the ℓ_i's.

Hence, by definition

$$p\{s|i\} = \sum_{\text{Group } i} p_{3\ell_1\ldots\ell_n}$$

$$p\{s|ii'\} = \sum_{\text{Group } i(i')} p_{3\ell_1\ldots\ell_n} + \sum_{\text{Group } i'(i)} p_{3\ell_1\ldots\ell_n}$$

where

$$p_{3\ell_1\ldots\ell_n} = p_{\ell_1}\left[p_{\ell_2}/(1-p_{\ell_1}) \ldots p_{\ell_n}/(1-p_{\ell_1}-\ldots-p_{\ell_{n-1}})\right].$$

Diagram of Scheme

Group	Sub-group	Permutation	
	Sub-group 1(2)	$\ell_1\ell_2\ldots$ ⋮ $\ell_1\ell_2\ldots$	(n-2)! permutations
	...	⋮	
Group (1)	Sub-group 1(i')	$\ell_1\ell_{i'}\ldots$ ⋮ $\ell_1\ell_{i'}\ldots$	(n-2)! permutations
	...		
	Sub-group 1(n)	$\ell_1\ell_n\ldots$ ⋮ $\ell_1\ell_n$	(n-2)! permutations
...	...		...
	Sub-group i(1)	$\ell_i\ell_1\ldots$ ⋮ $\ell_i\ell_1\ldots$	(n-2)! permutations
	...	⋮	
Group (i)	Sub-group i(i')	$\ell_i\ell_{i'}\ldots$ ⋮ $\ell_i\ell_{i'}\ldots$	(n-2)! permutations
	...	⋮	
	Sub-group i(n)	$\ell_i\ell_n\ldots$ ⋮ $\ell_i\ell_n\ldots$	(n-2)! permutations
...	...	...	...
	Sub-group n(1)	$\ell_n\ell_1\ldots$ ⋮ $\ell_n\ell_1\ldots$	(n-2)! permutations
	...	⋮	
Group (n)	Sub-group n(i')	$\ell_n\ell_{i'}\ldots$ ⋮ $\ell_n\ell_{i'}\ldots$	(n-2)! permutations
	...	⋮	
	Sub-group n(n-1)	$\ell_n\ell_{n-1}\ldots$ ⋮ $\ell_n\ell_{n-1}\ldots$	(n-2)! permutations

APPENDIX E

Derivations of $\epsilon v_2'(s')$ and $\epsilon v_2'^2(s')$ for Des Raj's method

Writing Des Raj's variance estimator as

$$n^2(n-1)\, v_2(s') = \sum_{\substack{i<i' \\ s\supset ii'}}^{n} \sum (t_i - t_{i'})^2$$

where s' denotes one of n! possible ordering of a given sample s and

$$t_1 = y_1/p_1$$

$$\cdots$$

$$t_i = \sum_{r=1}^{i-1} y_r + \left(1 - \sum_{r=1}^{i-1} p_r\right) y_i/p_i$$

$$\cdots$$

$$t_n = \sum_{r=1}^{n-1} y_r + \left(1 - \sum_{r=1}^{n-1} p_r\right) y_n/p_n$$

The 'trick' of this derivation is to write $(t_i - t_{i'})$ for a given s' as

$$(t_i - t_{i'}) = \sum_{j=1}^{n} C_{ii'j}\, y_j$$

where for $i < i'$

$$C_{ii'j} = \begin{cases} 0 & j < i \\ \left(\sum_{r=1}^{i} p_r - 1\right)/p_i & j = i \\ 1 & i < j < i' \\ \left(1 - \sum_{r=1}^{i-1} p_r\right)/p_i & j = i' \\ 0 & j > i' \end{cases}$$

and for $i > i'$ we have

$$C_{iij} = \begin{cases} 0 & j < i' \\ -C_{ii'i} & j = i' \\ -1 & i' < j < i \\ -C_{ii'i'} & j = i \\ 0 & j > i \end{cases}$$

Now, by letting $v_2'(s')$ be $v_2(s')$ under the assumption of the super population model we have

$$n^2(n-1) v_2'(s') = \sum_{\substack{i<i' \\ s'\supset ii'}}^{n} \sum \left[\sum_{j=1}^{n} C_{ii'j}\, e_j\right]^2$$

and

$$2n^2(n-1)\epsilon v_2'(s') = \sum_{\substack{i\neq i' \\ s'\supset ii'}}^{n} \sum \left[\sum_{j=1}^{n} C^2_{ii'j}\, x_j^g\right]$$

Similarly, $\epsilon v_2'^2(s')$ is obtained after considerable manipulation as

$$4n^4(n-1)^2\, \epsilon v_2'^2(s') = (2n^2(n-1)\,\epsilon v_2'(s'))^2$$

$$+ 4 \sum_{\substack{i\neq i' \\ s\supset ii'}}^{n} \sum \left[\sum_{j=1}^{n} C^2_{ii'j}\, x_j^g\right]^2$$

$$+ 4 \sum_{\substack{i\neq i'\neq i'' \\ s\supset ii'i''}}^{n} \sum \sum \left[\sum_{j=1}^{n} C_{ii'j} C_{ii''j}\, x_j^g\right]^2$$

$$+ 4 \sum_{\substack{i\neq i'\neq i'' \\ s'\supset ii'i''}}^{n} \sum \sum \left[\sum_{j=1}^{n} C_{ii'j} C_{i'i''j}\, x_j^g\right]^2$$

$$+ 2 \sum_{\substack{i\neq i'\neq i''\neq i''' \\ s\supset ii'i''i'''}}^{n} \sum \sum \sum \left[\sum_{j=1}^{n} C_{ii'j}\, C_{i''i'''j}\, x_j^g\right]^2 .$$

TABLE 3.1 Description of the natural population for n = 3

Pop. No.	Source	y	x	N	C.V.(y)	C.V.(x)	ρ
1	Horvitz Thompson (1952)	No. of house-holds	Eye-estimated no. of house-holds	20	0.44	0.40	.87
2	Des Raj (1965)	No. of house-holds	Eye-estimated no. of house-holds	20	0.44	0.41	.66
3	Rao (1963)	Corn acreage in 1960	Corn acreage in 1958	14	0.39	0.43	.93
4	Kish (1965)	No. of rented dwel-lings	Total no. of dwellings	15	1.37	1.06	.98
5	Cochran (1963)	Wt. of peaches	Eye-estimated wt. of peaches	10	0.19	0.17	.97
6	Hanurav (1967)	Popu-lation in 1960	Popula-tion in 1950	16	0.66	0.65	.99
7	Cochran (1963)	No. of per-sons per block	No. of rooms per block	10	0.15	0.14	.65
8	Cochran (1963)	No. of people in 1930	No. of people in 1920	20	0.85	0.93	.97
9	Cochran (1963)	No. of people in 1930	No. of people in 1920	20	0.71	0.82	.95
10	Sukhatme (1954)	No. of wheat A's in 1937	No. of wheat A's in 1936	20	0.76	0.74	.99
11	Sampford (1962)	Oat A's in 1957	Total A's in 1947	20	0.62	0.70	.83
12	Sukhatme (1954)	Wheat A's	No. of villages	20	0.59	0.51	.52
13	Yates (1960)	Volume of timber	Eye-estimate of volume	20	0.52	0.48	.50
14	Yates (1960)	No. of absen-tees	Total no. of persons	20	0.53	0.46	.67

TABLE 3.2. Percent gains in efficiency of the estimators over Sampford's estimator for n = 3 .

Pop. No.	Carroll-Hartley	Fellegi	Murthy	Des Raj	R.H.C.	Lahiri	With Rep.
1	+0	-0	-2	-3	-5	-22	-15
2	-0	-1	-0	-1	-3	-14	-13
3	-0	-2	3	1	+0	3	-14
4	-1	-8	1	-4	-3	-34	-24
5	-0	-1	1	-3	-2	2	-22
6	+0	-2	-0	-3	-7	-25	-19
7	+0	-1	1	-2	-1	6	-21
8	-0	-5	8	6	3	13	- 7
9	-0	-4	9	8	10	563	- 1
10	+0	-2	2	+0	-3	-10	-13
11	-0	-2	1	-0	-3	-16	-13
12	-0	-2	4	3	4	44	- 7
13	-0	-1	1	-1	-2	- 9	-12
14	-0	-1	2	1	1	10	- 9

TABLE 3.3. Percent gains in efficiency of the variance estimators over Sampford's variance estimator for n = 3 .

Pop. No.	Carroll-Hartley	Fellegi	Murthy	Des Raj	R.H.C.	Lahiri	With Rep.	Modified Lahiri
1	+0	1	-5	-6	-10	-100	-18	- 99
2	+0	-0	2	1	2	- 95	- 6	- 88
3	-0	-1	13	10	19	-100	- 7	-100
4	1	-0	45	32	47	- 99	22	- 99
5	-0	-1	6	4	9	-100	-17	-100
6	-0	-2	8	6	5	-100	- 9	-100
7	+0	-1	6	4	10	-100	-15	-100
8	+0	+0	15	11	21	- 99	2	- 99
9	+0	+0	12	10	22	- 34	4	71
10	+0	-2	22	22	42	-100	20	-100
11	1	-0	19	19	32	- 99	14	- 97
12	+0	-1	9	9	18	- 78	4	- 14
13	-0	-2	10	10	19	- 93	4	- 82
14	+0	-0	7	6	12	- 96	- 1	- 91

TABLE 3.4. Percent gains in average efficiency of the estimators over Sampford's estimator (under the super population model for g = 1.5, 1.75, and 2.00) for n = 3 .

(Natural Populations)

	g = 1.50			g = 1.75			g = 2.00		
Pop. No.	Murthy	Des Raj	R.H.C.	Murthy	Des Raj	R.H.C.	Murthy	Des Raj	R.H.C.
1	+0	-1	-2	+0	-1	-2	-0	-1	-2
2	+0	-1	-2	+0	-1	-2	-0	-1	-3
3	1	-2	-3	+0	-2	-4	-0	-3	-5
4	4	-2	-11	1	-6	-16	-2	-10	-20
5	+0	-4	-3	+0	-4	-3	-0	-4	-4
6	2	-1	-4	+0	-2	-6	-1	-3	-8
7	+0	-4	-3	+0	-4	-3	-0	-4	-3
8	3	+0	-5	1	-2	-8	-2	-5	-11
9	2	-0	-4	1	-2	-7	-1	-3	-9
10	1	-0	-4	1	-1	-6	-0	-2	-7
11	1	-0	-3	1	-1	-5	-0	-2	-6
12	1	-1	-2	+0	-1	-3	-0	-1	-4
13	1	-1	-2	+0	-1	-3	-0	-1	-3
14	1	-1	-2	+0	-1	-2	-0	-1	-3

TABLE 3.5. Percent gains in average efficiency of the variance estimators over Sampford's variance estimator (under the assumption of a super population model for $g = 1.50, 1.75$, and 2.00) for $n = 3$.

Natural Populations

	g = 1.50					g = 1.75					g = 2.00				
Pop. No.	Mur.	Des Raj	R.H.C.	Carroll Hartley	Fellegi	Mur.	Des Raj	R.H.C.	Carroll Hartley	Fellegi	Mur.	Des Raj	R.H.C.	Carroll Hartley	Fellegi
1	4	3	6	+0	-4	3	2	3	+0	-4	1	1	1	+0	-3
2	4	4	6	+0	-4	3	2	4	+0	-4	2	1	1	+0	-4
3	9	8	12	+0	-7	7	5	8	+0	-7	4	3	3	+0	-6
4	56	53	90	1	-16	50	45	62	1	-18	39	32	35	+0	-18
5	3	1	3	+0	-4	2	-0	2	+0	-4	1	-1	+0	+0	-4
6	17	16	25	+0	-10	14	12	17	+0	-10	10	7	9	+0	-9
7	2	-0	1	+0	-3	1	-1	1	+0	-3	1	-1	-0	-0	-3
8	31	28	44	+0	-14	29	26	37	+0	-15	25	20	24	+0	-15
9	20	19	30	+0	-10	17	15	21	+0	-10	12	9	10	+0	-10
10	13	13	21	+0	-7	9	7	10	+0	-7	4	3	2	+0	-6
11	13	12	20	+0	-8	10	9	12	+0	-7	6	4	4	+0	-7
12	6	6	9	+0	-5	4	3	5	+0	-5	2	1	1	+0	-4
13	6	5	8	+0	-5	4	3	4	+0	-4	2	1	1	+0	-4
14	5	5	8	+0	-5	4	3	4	+0	-4	2	1	1	+0	-4

TABLE 4.1. Description of the natural population for $n = 4$.

Pop. No.	Source	y	x	N	C.V.(y)	C.V.(x)	ρ
1	Horvitz & Thompson (1952)	No. of households	Eye-estimated no. of households	16	.40	.43	.91
2	Rao (1963)	Corn acreage in 1960	Corn acreage in 1958	14	.39	.43	.93
3	Kish (1965)	No. of rented dwelling units	Total no. of dwellings	15	1.37	1.06	.98
4	Cochran (1963)	Wt. of peaches	Eye-estimated wt. of peaches	10	.19	.17	.97
5	Hanurav (1967)	Population in 1960	Population in 1950	16	.66	.65	.99
6	Cochran (1963)	No. of persons per block	No. of rooms per block	10	.15	.14	.65
7	Cochran (1963)	No. of people in 1930	No. of people in 1920	12	.78	.95	.96
8	Sukhatme (1954)	No. of wheat A's in 1937	No. of wheat A's in 1936	13	.80	.76	.98
9	Sampford (1962)	Oats A's in 1957	Total A's in 1947	14	.65	.69	.75
10	Yates (1960)	Volume of timber	Eye-estimated volume of timber	11	.37	.45	.72

TABLE 4.2. Percent gains in efficiency of the estimators over Sampford's estimator for n = 4.

Natural Populations						
Pop. No.	Carroll-Hartley	Murthy	Des Raj	R.H.C.	Lahiri	With Repl.
1	-0	-0	- 4	- 5	-16	-24
2	-0	4	+ 0	- 1	6	-21
3	-1	-4	-14	-24	-44	-39
4	-0	2	- 6	- 5	3	-32
5	+0	-1	- 6	-10	-31	-28
6	+0	2	- 5	- 3	8	-31
7	-0	33	25	33	849	- 3
8	+0	+0	-10	-18	-34	-37
9	+0	-2	- 9	-15	-30	-33
10	-0	4	- 4	- 4	- 0	-30

TABLE 4.3. Percent gains in efficiency of the variance estimators over Sampford's variance estimator for n = 4 .

Pop. No.	Carroll-Hartley	Murthy	Des Raj	R.H.C.	Lahiri	With Repl.	Modified Lahiri
1	+0	4	+0	2	-100	-16	-100
2	-0	19	12	27	-100	-10	-100
3	+1	83	56	140	- 99	52	- 99
4	-0	10	3	12	-100	-23	-100
5	-0	13	8	8	-100	-12	-100
6	-0	11	4	15	-100	-21	-100
7	+1	43	23	75	- 75	13	- 33
8	+2	120	128	242	-100	135	-100
9	+0	96	100	153	- 98	92	- 96
10	-0	21	13	31	- 99	- 5	- 99

TABLE 4.4 Description of the natural populations for n = 4 under the super population model.

Natural Populations									
	g = 1.50			g = 1.75			g = 2.00		
Pop. No.	Murthy	Des Raj	R.H.C.	Murthy	Des Raj	R.H.C.	Murthy	Des Raj	R.H.C.
1	1	- 9	-10	+0	-10	-11	-1	-11	-13
2	1	- 8	- 8	-0	- 9	- 9	-1	-10	-11
3	6	-15	-25	-2	-22	-33	-9	-30	-40
4	+0	- 8	- 7	-0	- 8	- 7	-0	- 8	- 8
5	4	-12	-19	-2	-18	-26	-8	-24	-32
6	+0	- 8	- 7	+0	- 8	- 7	-0	- 8	- 7
7	1	-10	-13	-0	-11	-15	-1	-13	-17
8	4	-13	-21	-1	-17	-26	-4	-21	-30
9	3	-11	-16	-1	-15	-21	-6	-20	-26
10	2	- 9	-12	-0	-11	-14	-2	-13	-17

TABLE 4.5 Percent gains in average efficiency of the estimators over Sampford's estimator (using the super population model for g = 1.50, 1.75, and 2.00) for n = 4.

Pop. No.	Source	y	x	N	C.V.(y)	C.V.(x)	ρ
1	Horvitz & Thomp. (1952)	No. of households	Eye-estimated no. of households	10	.41	.37	.89
2	Rao (1963)	Corn A's in 1960	Corn A's in 1958	10	.23	.30	.81
3	Kish (1965)	No. of rented dwelling units	Total no. of dwelling units	10*	1.41	.82	.93
4	Cochran (1963)	Wt. of peaches	Eye-estimated wt. of peaches	10	.19	.17	.97
5	Hanurav (1967)	Population in 1960	Population in 1950	10	.75	.73	.99
6	Cochran (1963)	No. of persons per block	No. of rooms per block	10	.15	.14	.65
7	Cochran (1963)	No. of people in 1930	No. of people in 1920	10	.31	.47	.34
8	Suk. (1954)	No. of wheat A's in 1937	No. of wheat A's in 1936	10	.75	.69	.98
9	Sampford (1962)	Oats A's in 1957	Total A's in 1947	10	.58	.64	.91
10	Yates (1960)	Volume of timber	Eye-estimated vol. of timber	10	.39	.47	.72

* one x-value changed so that $\pi_i \leq nP_i$ for i = 1,...,10

TABLE 4.6. Percent gains in average efficiency of the variance estimators over Sampford's variance estimator (using the super population model for g = 1.50, 1.75, and 2.00) for n = 4.

Natural Populations												
	g = 1.50				g = 1.75				g = 2.00			
Pop. No.	Carroll Hartley	Murthy	Des Raj	R.H.C.	Carroll Hartley	Murthy	Des Raj	R.H.C.	Carroll Hartley	Murthy	Des Raj	R.H.C.
1	+0	23	17	28	+0	19	12	19	+0	14	7	10
2	+0	15	10	17	+0	13	7	12	+0	10	4	7
3	3	164	159	257	2	163	151	201	1	149	129	145
4	+0	5	-0	3	+0	4	-1	2	+0	3	-2	+0
5	+0	111	101	158	+0	112	98	134	-0	106	88	103
6	+0	3	-2	1	+0	2	-3	+0	+0	2	-3	-1
7	+0	34	29	45	+0	23	15	22	+0	17	9	11
8	2	90	85	135	1	75	64	84	1	60	47	52
9	+0	74	64	101	+0	73	61	85	+0	68	54	65
10	+0	40	33	50	+0	35	27	38	+0	28	20	25